that time derivatives are now zero only for an observer moving with the char is forgotten), and solution of the resulting equation with boundary conditions (7) and (8) leads Chen to deduce a linear variation of temperature in the virgin material!

5) The model of ablation presented considers both a charring process and a surface-mass removal process. However, Chen does not explicitly make any statement regarding the mathematical model for surface-mass loss. It should be pointed out that there are four unknowns in the problem: a) the temperature distribution in the char-gas layer, b) the temperature distribution in the virgin-material layer, c) the thickness of the ablating layer s, representing the amount of surface matter removed, and d) the thickness of the char-gas layer δ_c. (Of course, once s and δ_c are known, the thickness of the virgin-material layer δ_s automatically follows.)

Unknowns a) and b) each are described by a partial differential equation, two boundary conditions, and an initial condition. Unknowns c) and d) each have an initial condition (presumably, $s(0) = \delta_c(0) = 0$). However, they have only one equation between them, viz., Eq. (10). It is therefore necessary for the solution of this problem to have another equation involving s and/or δ_c. For the steady-state case, such an equation is offered by the implicit relation:

$$(d\delta_c/d\theta) = 0$$

For the transient case, such a relationship is also necessary. This could be provided by the (apparent) assumption mentioned under item 3 preceding,

$$s(\theta) = \delta_c(\theta)$$

This relationship does not seem to have any justification, either theoretical or experimental. However, without this assumption, Chen's stated solution would not be a solution at all.

References

[1] Chen, N. H., "Simplified solutions for ablation in a finite slab," AIAA J. **3**, 1148–1149 (1965).

[2] Barriault, R. J. and Yos, J., "Analysis of the ablation of plastic heat shields that form a charred surface layer," ARS J. **30**, 823–829 (1960).

Reply by Author to D. B. Adarkar

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ATTENTION should be called to the fact that ablation is a complicated phenomenon. It deals with heat transfer (conduction, convection, and radiation), mass transfer (diffusion), chemical reaction (combustion), and kinetics. The products of the combustion are char and multicomponent gaseous mixtures. In addition, the heat input from the environment to initiate the ablation in a rocket motor also is concerned with multicomponent exhaust gaseous mixtures of which the composition and the convective heat-transfer coefficient still are evaluated approximately. Because of this complex and approximate nature, it is definitely impossible to obtain a closed form solution to account for all these effects. Hence, for the past years, a great number of articles (probably over a hundred) have appeared on the basis of simplified assumptions.

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With these simplified assumptions in mind, it is evident that the solution is not exact but approximate. The purpose of my previous note[1] is to obtain a simple and approximate analytical expression with simplified assumptions.

To fulfill this intention, the temperature gradient in the virgin material layer was assumed negligible in order to reduce the boundary condition, Eq. (10), to a simpler form, Eq. (11). Consequently, the final solution would be simpler. The validity of this assumption easily can be seen by the fact that, for a thin virgin-material layer, the resistance to the heat flow is comparatively small, so that the temperature gradient cannot be great. It is always desired to have economical but adequate thickness of ablative material as insulator in a rocket motor. This implies that the insulation should be designed as thin as possible, such that a large part of the insulation will undergo ablation to absorb a large part of the heat from the environment. Only a small portion of the heat transfers to the thin layer of the virgin material. Furthermore, the firing of a rocket motor is usually short. For all these reasons, it appears that this assumption of negligible temperature gradient in the virgin material layer is very reasonable for obtaining simplified analytical solutions.

To repeat, the prime objective of my note is to acquire an analytical solution through simplified assumptions. In order to save space in the publication, no attention has been paid to the consistency of the formulation of the boundary conditions and the governing partial differential equations. As long as the idea is understood, the objective is accomplished. The use of the coordinate x in Eq. (4) means that the initial temperature in the char layer and the virgin-material layer before ablation is T_i. Its use appears to be more general. The use of Eq. (1) to represent the virgin-material layer also is clear. For saving space, it is unnecessary to change coordinate, provided that the correct values and boundary conditions are applied.

It is incorrect to write the expression

$$L\rho_s[(ds/d\theta) + (d\delta_c/d\theta)]$$

as indicated by Adarkar. The term δ_c is not the dependent variable. The $ds/d\theta$ term applies not only to the ablating layer, but also to the char-gas layer as well. Hence, this term is sufficient, and the term $d\delta_c/d\theta$ is wrong. Even if the last term is changed to $d\xi/d\theta$, it is still unnecessary to have it. Hence, the formulation of Eq. (10) and Eq. (11) is still correct inasmuch as the rate of the heat penetration is still $ds/d\theta$ and the temperature gradient in the virgin-material layer is assumed negligible.

Equation (22) does satisfy the boundary condition Eq. (6) because, at δ_c, Eq. (19) really becomes (but not the unlikely case as indicated by Adarkar)

$$s = \delta_c = 2\lambda(\alpha\theta)^{1/2}$$

Equation (23) comes from Eq. (22) and Eq. (9). The latter is the boundary condition at which the heat flows from the environment. Some investigators have pointed out that the heat flux as $H(\theta)$ could be a function of time. For this reason, this $H(\theta)$ is equal to $h(T_g - T_0)$ in my note. In order to have approximate solution in our problem, this $h(T_g - T_0)$ also will be proportional to $1/\theta^{1/2}$, so that the time term can be cancelled out. For the sake of saving space, this explanation did not appear in my note because it is already understood by those who have had sufficient knowledge on ablation.

It is now necessary to explain the proposed model and the formulation. The proposed model represents the ablation with the removal of the ablating layer. Because of this removal, in order to calculate the temperature gradient in the char-gas layer, it is proposed to transform the independent variable x in Eq. (1) to another independent variable ξ by Eq. (2). However, in the calculation of the temperature in the virgin-material layer, it is unnecessary to make this transformation, because the original Eq. (1) can be used much more easily, provided that the correct values are used. By this consideration, it is unnecessary to indicate the stationary and moving frame as done by Adarkar. In conclusion, to my knowledge the problem in my previous note is mathematically well posed.

Reference

[1] Chen, N. H., "Simplified solutions for ablation in a finite slab," AIAA J. 3, 1148–1149 (1965).

Comment on "Validity of Integral Methods in MHD Boundary-Layer Analysis"

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Heywood and Moffatt[1] have given a detailed and lucid discussion on the difficulties involved in applying the so-called integral methods to MHD problems. Here, we wish to supplement their work by emphasizing the fact that the difficulties are inherent in nature, and much more deeply rooted than usually suspected. We wish to exemplify this by the following observation.

Consider the MHD Rayleigh problem[2] governed by the following system:

$$\begin{cases} \partial u/\partial t = \nu(\partial^2 u/\partial y^2) + a^2(\partial b/\partial y) \\ \partial b/\partial t = \eta(\partial^2 b/\partial y^2) + (\partial u/\partial y) \\ t = 0: \quad b = u = 0 \\ y = 0: \quad u = 1, \quad b = 0 \quad \text{(perfectly insulating)} \\ y \to \infty: \quad u \to 0, \quad b \to 0 \end{cases}$$

It is not necessary to explain the meaning of the symbols used here, except to indicate that $a^2(\partial b/\partial y)$ and $\partial u/\partial y$ represent the interactions between the magnetic and the flow fields. Now if we integrate this system with respect to y from 0 to ∞, the resulting integral relations are

$$\frac{d}{dt}\int_0^\infty u\,dy = -\nu\left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$\frac{d}{dt}\int_0^\infty b\,dy = -\eta\left.\frac{\partial b}{\partial y}\right|_{y=0} - 1$$

The influence of the magnetic field on the flow field is lost completely, whereas that of the flow field on the magnetic field is over-simplified to a constant term 1. It is obvious that regardless of which trial curves one uses for u and b, the result will not be acceptable.†

It is interesting, however, to notice that the same problem, if the induced magnetic field is neglected because of small viscous diffusivity compared to the magnetic diffusivity, is governed by one single equation

$$\partial u/\partial t = \nu(\partial^2 u/\partial y^2) - cu$$

which yields the integral relation

$$\frac{d}{dt}\int_0^\infty u\,dy = -\nu\left.\frac{\partial u}{\partial y}\right|_{y=0} - c\int_0^\infty u\,dy$$

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† The two integral relations are correct, of course. It is only incorrect (not even as an approximation) here to try to obtain local behavior from them. In other words, it is true that the net effect of the interaction on the fluid as a whole is represented by the two constants 0 and 1; locally, however, this is simplifying the matter too much.

In this equation, the interaction is decently represented. As a matter of fact, a trial solution of the form

$$u = \operatorname{erfc}\left[\frac{y}{2(\nu)^{1/2}\delta(t)}\right]$$

yields results that check very well with the exact solution for all y values at small times. For large time, the check is very good near the wall, as expected.

References

[1] Heywood, J. B. and Moffat, W. C., "Validity of integral methods in MHD boundary-layer analyses," AIAA J. 3, 1565–1567 (1965).

[2] Bryson, A. E. and Rościszewski, J., "Influence of viscosity, fluid conductivity and wall conductivity in the magnetohydrodynamic Rayleigh problems," Phys. Fluids 5, 175–183 (1962).

Comments on the Analysis of Free Vibration of Rotationally Symmetric Shells

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In a recent paper,[1] Stodola's iterative method of finding eigenvalues has been proposed for the analysis of free vibration of shells of revolution. This method is being offered as an alternate to the multisegment free vibration method employed earlier.[2] The writer would like to make some comments on the merits of these two methods.

First of all, the multisegment free vibration method employed in Ref. 2 is not an "iterative" method in the same sense as Stodola's method. According to the multisegment method, the frequency equation of an arbitrary shell of revolution is obtained in the form of a determinant of a (4 × 4) matrix whose elements, for a given frequency, are determined by means of direct numerical integration of the differential equations. The roots of the frequency equation are found in the same way as the roots of any transcendental algebraic equation. In practice, the finding of the roots is a relatively simple matter, and it involves a systematic evaluation of the frequency equation at selected points within a given frequency interval. As soon as a sign change in the determinant is detected, the natural frequency is determined as accurately as desired by means of inverse interpolation. The point is that no "convergence" is involved in the sense of convergence of an assumed solution toward the actual solution, which is the basis of the usual "iterative" methods, such as Stodola's method.

As is well known,[3] Stodola's method (sometimes called the method of Stodola and Vianello) has been of great practical value in vibration and stability problems of beams. The method starts with a rough estimate of the deflection curve of the beam, from which a better estimate is obtained. It can be proved (Ref. 3, p. 201) that this iteration process is convergent for the lowest eigenvalue. Higher eigenvalues can be obtained by subtracting out from the assumed solution all preceding eigenfunctions.

In the opinion of the writer, the only reason for wanting to apply Stodola's method to shells of revolution is that instead of eight homogeneous solutions, which are needed to find one value of the frequency determinant by the multisegment method, only one particular solution per iteration need be calculated. However, whereas in the multisegment method

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